Coursework Exercie 4 of Physics 2

Electric conductors and capacitors

To calculate the capacitance of a capacitor, follow the following steps:
1-Calculate the field between the armatures (using the Gauss theorem)
2-Deduce the potential difference between the conductors ($\vec{E}=-\overrightarrow{grad}(v)$)
3-Calculate the ratio C=Q/(V1-V2)

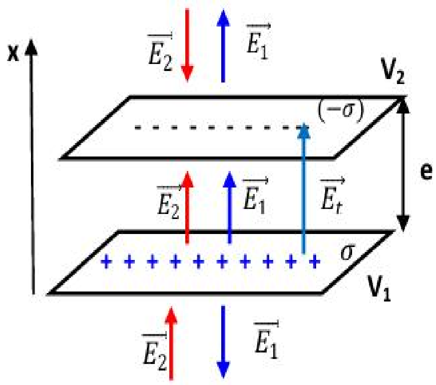
Exercise 1: parallel-plane capacitor

$\vec{E}_1$ ⊥ to the plane and exiting the plane (positive charge)
$\vec{E}_2$ ⊥ to the plane and entering the plane (negative charge)"

Outside the plates $\vec{E}_1 + \vec{E}_2 = \vec{0}$

Between the plates $\vec{E}_{total} = \vec{E}_1 + \vec{E}_2$

By applying Gauss's theorem $\Phi = \oiint_S \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{ins}}{\varepsilon_0}$



Let us take as the Gaussian surface of a cylinder with an axis perpendicular to the plane (+)

$$\Phi(\vec{E}_1) = \Phi_{S_1}(\vec{E}_1) + \Phi_{S_2}(\vec{E}_1) + \Phi_{S_3}(\vec{E}_1)$$

$$\Phi(\vec{E}_1) = \oiint_{S_1} \vec{E}_1\overrightarrow{dS_1} + \oiint_{S_2} \vec{E}_1.\overrightarrow{dS_2} + \oiint_{S_3} \vec{E}_1\overrightarrow{dS_3}$$

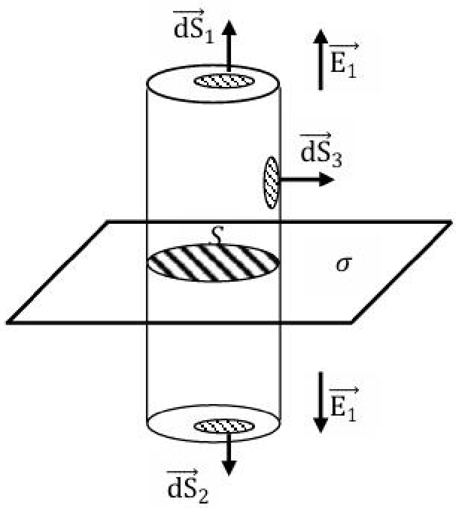
The field $\vec{E}$ is perpendicular to $\overrightarrow{dS_3}$ so $\Phi_{S_3} = 0$

The field $\vec{E}$ is paralell to $\overrightarrow{dS_1}$ and $\overrightarrow{dS_2}$

$$\Phi(\vec{E}_1) = \Phi_{S_1} + \Phi_{S_2} = ES_1 + ES_2 = 2ES_1 = 2ES \quad (S_1 = S_2 = S)$$

So the charge contained in the surface of *Gauss* is

$$Q = \iint_s dq = \sigma \iint_s dS = \sigma S$$



We apply Gauss' theorem:

$$\Phi(\vec{E}_1) = 2E_1 S = \frac{\sigma S}{\varepsilon_0} \Rightarrow \boxed{E_1 = \frac{\sigma}{2\varepsilon_0}}$$

Same result for the plane (-) $\boxed{E_2 = \frac{\sigma}{2\varepsilon_0}}$

$$E_{total} = E_1 + E_2 \Rightarrow \boxed{E = \frac{\sigma}{\varepsilon_0}}$$

2) $\vec{E}=-\overrightarrow{grad}(v) \Rightarrow E = \frac{-dV}{dr} \Rightarrow \int_{V_1}^{V_2} dV = -\int_0^e E.dx$

$$V_2 - V_1 = -\frac{\sigma}{\varepsilon_0}.e \Rightarrow V_1 - V_2 = \frac{\sigma e}{\varepsilon_0}$$

$$C = \frac{Q}{V_1 - V_2} = \frac{\sigma S}{\frac{\sigma e}{\varepsilon_0}} = \frac{S\,\varepsilon_0}{e} \Rightarrow \boxed{C = \frac{S\varepsilon_0}{e}}$$

We notice that the capacitance does not depend on the charge or the potential, it depends on the dimensions of the capacitor and the medium in which it is placed (here the vacuum ε_0)

Exercise 2: cylindrical capacitor

By applying Gauss's theorem. Let us take as Gaussian surface a cylinder of height h and radius r (R_1< r < R_2). Due to symmetry, E is radial and constant in the Gaussian surface.

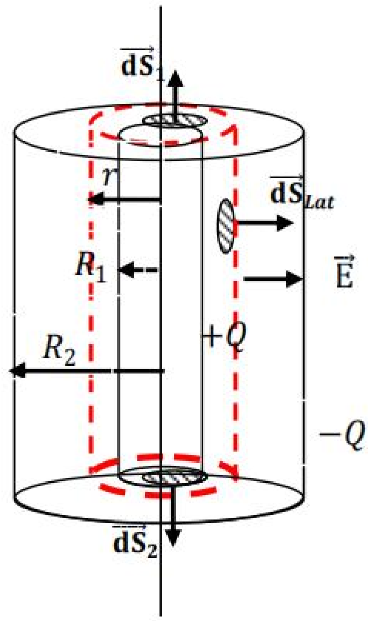
$$\Phi(\vec{E}) = \Phi_{S_1}(\vec{E}) + \Phi_{S_2}(\vec{E}) + \Phi_{S_3}(\vec{E}) \quad = \oiint_{S_1} \vec{E}.\overrightarrow{dS_1} + \oiint_{S_2} \vec{E}.\overrightarrow{dS_2} + \oiint_{S_3} \vec{E}.\overrightarrow{dS_3}$$

Field $\vec{E}$ is perpendicular to S_1 and $S_2 \Rightarrow \ \Phi_{S_1} = \Phi_{S_2} = 0$

Also, the field $\vec{E}$ is parallel to the normal of the lateral surface $\overrightarrow{S_3}$

$$\Phi_{S_3}(\vec{E}) = \oiint_{S_3} E.dS_3 = E.S_3 = E.2\pi.r.h$$

$$\Phi(\vec{E}) = E.S_3 = E.2\pi.r.h$$



The total charge contained in the Gaussian surface is Q

$$\Phi(\vec{E}) = E.2\pi.r.h = \frac{Q}{\varepsilon_0} \Rightarrow \boxed{E = \frac{Q}{2\pi.r.h\varepsilon_0}}$$

2) $\vec{E} = -\overrightarrow{grad}(v) \Rightarrow E = \frac{-dV}{dx} \Rightarrow \int_{V_1}^{V_2} dV = -\int_{R_1}^{R_2} E.dr$

$$V_2 - V_1 = -\int_{R_1}^{R_2} \frac{Q}{2\pi.r.h\varepsilon_0}.dr = -\frac{Q}{2\pi h\varepsilon_0} ln\,(r)|_{R_1}^{R_2}$$

$$\Rightarrow V_1 - V_2 = \frac{Q}{2\pi h\varepsilon_0} \ln\frac{R_2}{R_1}$$

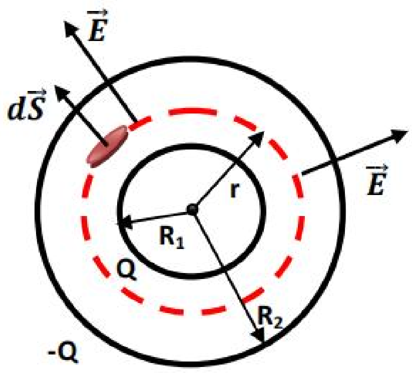
$$C = \frac{Q}{V_1 - V_2} = \frac{Q}{\frac{Q}{2\pi h\varepsilon_0}\ln\frac{R_2}{R_1}} = \frac{2\pi h\varepsilon_0}{\ln\frac{R_2}{R_1}} \Rightarrow \ C = 2\pi h\varepsilon_0 \ln\frac{R_1}{R_2}$$

Exercise 3: spherical capacitor

By applying Gauss' Theorem, let us calculate the electrostatic field created by a sphere with center O and radius $R_1 < r < R_2$.

For reasons of symmetry, the vector $\vec{E}$ is radial and has the same modulus on the Gaussian surface, the flow leaving this sphere is:

$$\Phi(\vec{E}) = \oiint_{SG} \vec{E}.\overrightarrow{dS} = E.SG = E4\pi r^2$$



The total charge contained in the Gaussian surface is Q

By applying Gauss' Theorem,

$$\Phi(\vec{E}) = E4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow \boxed{E = \frac{Q}{4\pi r^2\varepsilon_0}}$$

$$\vec{E} = -\overrightarrow{grad}(v) \Rightarrow E = \frac{-dV}{dr} \Rightarrow \int_{V_1}^{V_2} dV = -\int_{R_1}^{R_2} E.dr$$

$$V_2 - V_1 = -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1}\right) \Rightarrow V_1 - V_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\boxed{C = \frac{Q}{V_1 - V_2} = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}}$$

2) Calculation of capacity C if R_2 tends towards R_1

If $R_2 \rightarrow R_1 \Rightarrow \begin{cases} R_2 - R_1 \approx e \\ R_1 R_2 \approx R^2 \end{cases} \Rightarrow \boxed{C = \frac{4\pi\varepsilon_0 R^2}{e}}$

$S = 4\pi R^2$

So $\boxed{C = \frac{S\varepsilon_0}{e}}$ **similar to that of a planar capacitor**