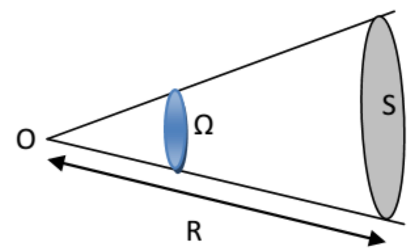
Theorem Gauss's

Introduction

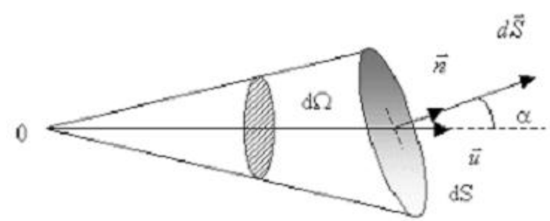
Gauss' theorem allows you to quickly calculate the electric field created by symmetrical charge distributions. First, we must define the notions of the Solid Angle and the flow of the electric field through a surface.

- angle solid of Concept

We saw in the previous study plane angles. But when it comes to spatial geometry, we find the solid angle. The solid angle is an “angle” in space, consider a sphere with center O and radius r.



We define the solid angle Ω under which we see a surface (S), from a point O, contained in a cone with vertex O.



- $d\Omega = \frac{\overrightarrow{ds}.\vec{u}}{R^2} = \frac{ds.\vec{n}.\vec{u}}{R^2} = \frac{ds.cos\alpha}{R^2}$
- Noticed: If $\vec{n}$ and $\vec{u}$ are parallel, therefore $cos\alpha = 1$, and consequently the solid angle is equal to: $d\Omega = \frac{ds}{R^2} => \Omega = \frac{s}{R^2}$

Concept of flow of the electric field through any surface

The flow of the field $\vec{E}(M)$ created at one point M by a charge distribution Q through a closed surface (S) is defined by:

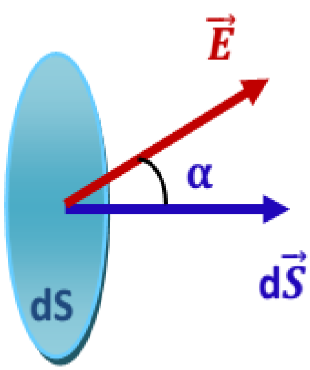
$$\Phi_S = \oiint_S \vec{E}(M)\,\overrightarrow{ds}(M)$$

- With $\overrightarrow{ds}$ elementary surface vector: $\overrightarrow{ds} = ds.\vec{n}$ And $\vec{n}$ unit vector

Theorem Gauss's

The flow of the field $\vec{E}$ through a closed surface created by a distribution of charges is equal to the algebraic sum of the charges present inside this surface (S_G) divided by ε_0

$$\Phi_S = \oiint_S \vec{E}(M)\,\overrightarrow{ds}(M) = \frac{\sum Q}{\varepsilon_0}$$



* The relationship between solid angle and electric flux:

The electric field produced by a point charge q at a distance from the charge is $E = \frac{kq}{r^2}$

Flow through an elementary surface dS located at distance r of charge q is :

$$\Phi_S = \oiint_S \vec{E}.\overrightarrow{ds} = \oiint_S K\frac{q}{r^2}ds.\cos\alpha = \oiint_S K.q.d\Omega$$

Remarks

1. $\overrightarrow{ds}$ surface vector ⊥ on the surface is directed from the inside to the outside.
2. Gauss' theorem applies to charge distributions where there is symmetry.
3. $\vec{E} \perp S_G$ (Gaussian surface) and E constant on S_G where S_G is an equipotential surface.

- General calculation method:

- Find a closed surface passing through the point M where you want to calculate the field.
- Write the definition of the flow $\Phi_S = \oiint_S \vec{E}.\overrightarrow{ds}$
- Apply Gauss' theorem after calculating the algebraic charge inside the surface.

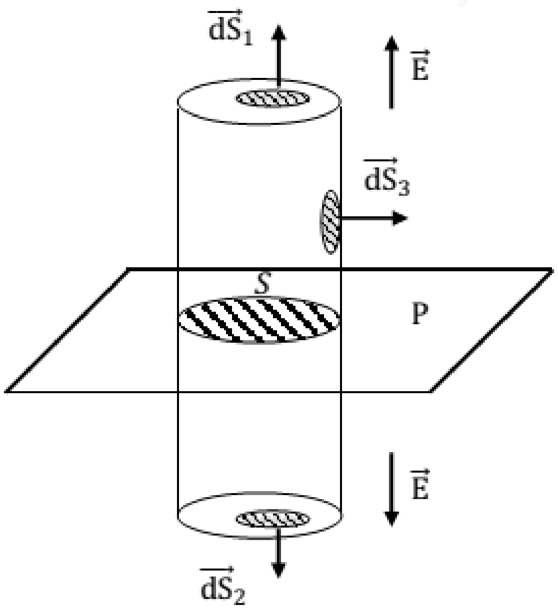
Example 1

- Calculation of the electrostatic field created by a wire of infinite length and constant linear density λ positive by application of Gauss' theorem.

- By application of Gauss's Theorem calculate the electrostatic field created by this distribution at a point located at distance x from the wire.

SOLUTION

- Let us take as a closed surface (surface of *Gauss*), a cylinder of radius x and length l and axis the infinite wire.



- By reason of symmetry the field $\overrightarrow{E}$ is radial (carried by ox)

$$\Phi(\vec{E}) = \Phi_{S_1}(\vec{E}) + \Phi_{S_2}(\vec{E}) + \Phi_{S_3}(\vec{E})$$

$$\Phi(\vec{E}) = \oiint_{S_1} \vec{E}.\overrightarrow{dS_1} + \oiint_{S_2} \vec{E}.\overrightarrow{dS_2} + \oiint_{S_3} \vec{E}.\overrightarrow{dS_3}$$

- Field $\vec{E}$ is perpendicular to the normal at any point

on both bases $\overrightarrow{S_1}$ And $\overrightarrow{S_2}$

- So : $\vec{E}.\overrightarrow{dS_1} = \vec{E}.\overrightarrow{dS_2} = 0 \Rightarrow \Phi_{S_1}(\vec{E}) = \Phi_{S_2}(\vec{E}) = 0$
- Also, the field $\vec{E}$ is parallel to the normal of the lateral surface $\overrightarrow{S_3}$
- $\Phi_{S_3}(\vec{E}) = \oiint_{S_3} E.dS_3 = E.S_3 = E.2\pi.x.l$

- 2)The total charge contained in the surface of *Gauss* is .

$$Q = \int dq = \lambda \int dl = \lambda . dl$$

- By applying the theorem of *Gauss*

$$E.2\pi.x.l = \frac{\lambda l}{\varepsilon_0}$$

Example 2

a- Calculation of the electrostatic field created at any point M in the space of an infinite plane(P) of uniform surface density σ positive by application of Gauss' theorem.

b-Calculate the electrostatic field generated by two infinite perpendicular planes, and with respective charge densities σ and 2σ.

Solution

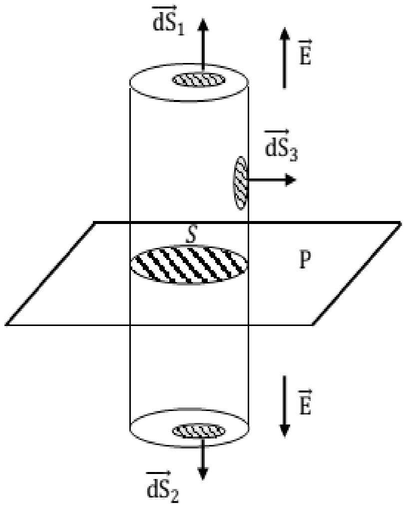
- 1)-Let us take as the surface of *Gauss* a cylinder with an axis perpendicular to the plane. Because of symmetry the field $\vec{E}$ is perpendicular to the plane (P) The flow of the vector $\vec{E}$ emerging from the surface of *Gauss* is
- $\Phi(\vec{E}) = \Phi_{S_1}(\vec{E}) + \Phi_{S_2}(\vec{E}) + \Phi_{S_3}(\vec{E})$
- $\Phi(\vec{E}) = \oiint_{S_1} \vec{E}.\overrightarrow{dS_1} + \oiint_{S_2} \vec{E}.\overrightarrow{dS_2} + \oiint_{S_3} \vec{E}.\overrightarrow{dS_3}$

Field $\vec{E}$ is perpendicular to the normal of the lateral surface $\overrightarrow{S_3} \Rightarrow \Phi_{S_3} = 0$

- On the other hand, we have nothing to do with the two bases S_1And S_2 ,

field $\vec{E}$ is parallel to the normal so

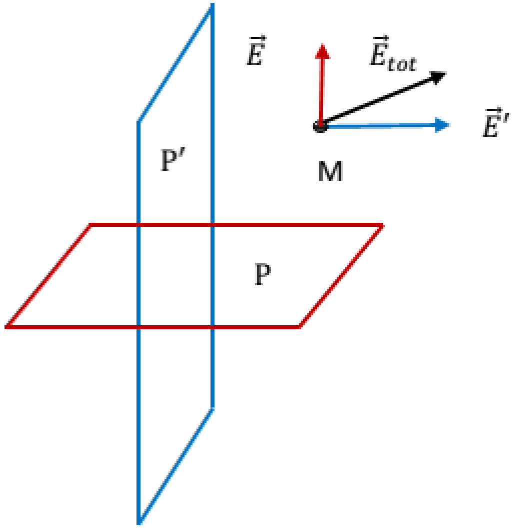
$$\Phi(\vec{E}) = \Phi_{S_1} + \Phi_{S_2} = \mathrm{E}S_1 + \mathrm{E}S_2 = 2ES_1 = 2ES \quad (S_1 = S_2 = S)$$



- The charge contained in the surface of *Gauss* is
- $Q = \iint_S dq = \sigma \iint_S dS = \sigma S$ withe $(S_1 = S_2 = S)$
- We apply Gauss' theorem:
- $\Phi(\vec{E}) = 2ES = \dfrac{\sigma S}{\varepsilon_0} \Rightarrow E = \dfrac{\sigma}{2\varepsilon_0}$

- 2)-By analogy with question 1, the field $\overrightarrow{E'}$ created by the plan$(P')$is

$$E' = \frac{\sigma}{\varepsilon_0}$$

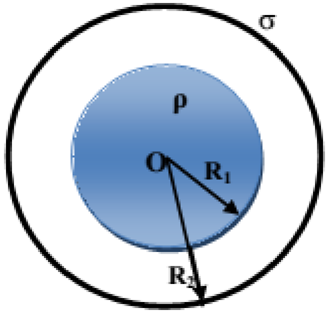


- Field $\vec{E}$ resulting is then:
- $\vec{E}_{tot} = \vec{E} + \vec{E}'$
- $E_{tot} = \sqrt{E^2 + E'^2}$

$$= \sqrt{(\frac{\sigma}{2\varepsilon_0})^2 + (\frac{\sigma}{\varepsilon_0})^2} = \frac{\sqrt{5}}{2}\frac{\sigma}{\varepsilon_0} \Rightarrow E_{tot} = \frac{\sqrt{5}}{2}\frac{\sigma}{\varepsilon_0}$$

Example 3

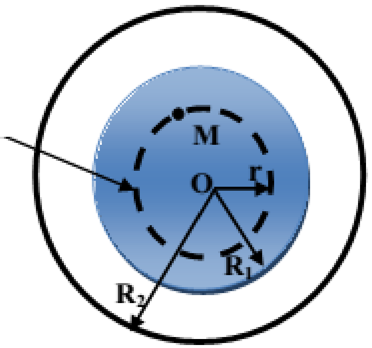
- Consider two concentric spheres of radius R_1 and R_2 ($R_1 < R_2$). The outer sphere of radius R_2 is charged with a constant and positive surface density σ, as for the interior sphere of radius R_1 it is charged with a volume density ρ constant and positive. Using Gauss' theorem, determine the electrostatic field $E(r)$ at any point in space.



Solution

Given the symmetry of the problem, the field is radial.

- *a) 1st case:* $r < R_1$



- The flux Φ leaving the Gaussian sphere is:

$$\Phi = E_1(r)\, S_G = E_1(r)\, 4\pi r^2 \; (S_G = S_{Gauss})$$

- The internal charge of the Gaussian sphere is:

$$\sum q_{int} = \int_0^r \rho\, dv = \int_0^r \rho 4\pi r^2 dr = \frac{4\rho\pi r^3}{3}$$

- *We will therefore have:*

$$\Phi = E_1 S = \frac{\sum q_{int}}{\varepsilon_0} \Rightarrow E_1 4\pi r^2 = \frac{4\rho\pi r^3}{3\varepsilon_0} \Rightarrow E_1 = \frac{\rho r}{3\varepsilon_0}$$

b) <u>**2^{nd} case**</u>**:** $R_1 < \mathbf{r} < R_2$

- $\Phi = E_{_2}\,(r)\, S_{_G} = E_{_2}\,(r)\, 4\pi r^{\,2}\ (\,S_{_G} = S_{_G}auss)$

- The internal charge of the Gaussian sphere is:

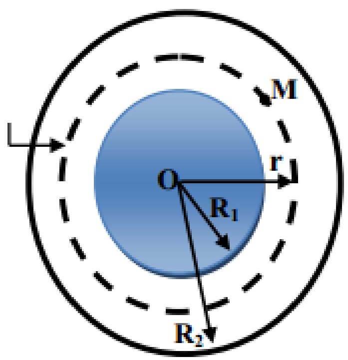
- $\sum q_{int} = \int_0^{R_{_1}} \rho dv\ = \int_0^{R_{_1}} \rho 4\pi r^{\,2} dr = \frac{4\rho\pi_{R^3_{_1}}}{3}$

He comes :

- $\Phi = E_{_2}\ S = \frac{\sum q_{int}}{\varepsilon_0}$

- $E_{_2}\ = \frac{\rho_{R^3_{_1}}}{3\varepsilon_0 r^{\,2}}$

- $\Rightarrow E_2 4\pi r^{\,2} = \frac{4\rho\pi R^3_{_1}}{3\varepsilon_0}$



c) 3rd case: r >R_2

$$\Phi = E_3(r)\, S_G = E_3(r)\, 4\pi r^2 \;(S_G = S_{Gauss})$$

- The internal charge of the Gaussian sphere is:

$$\sum q_{int} = q_{R_1} + q_{R_2} = \frac{4\rho\pi R_1^3}{3} + \int_0^{R_2} \sigma dS \; = \frac{4\rho\pi R_1^3}{3} + \sigma \int_0^{R_2} 8\pi r dr \; = \frac{4\rho\pi R_1^3}{3} + 4\pi\sigma R_2^2$$

- Eventually :

$$E_3(r)\, 4\pi r^2 = \frac{4\rho\pi R_1^3}{3\varepsilon_0} + \frac{4\rho\pi R_2^2}{\varepsilon_0} \Rightarrow E_3(r) \; = \frac{\rho R_1^3 + 3\sigma R_2^2}{3\varepsilon_0 r^2}$$