Coursework Exercise 2 of Physics 2

Continuous distribution and electric dipole

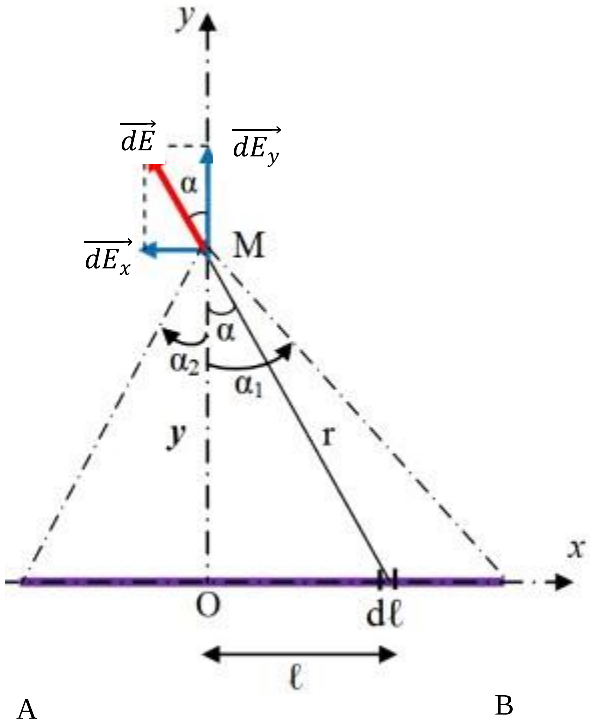
Exercise 1:

1°)a) Let us calculate the field **EM** created at a point M of the axis OY:

The elementary field dE due to an element of length dℓ with charge dq = λ dℓ has the expression

$$\overrightarrow{dE} = \overrightarrow{dE_x} + \overrightarrow{dE_y} \quad \text{where} \quad \begin{cases} \overrightarrow{dE_x} = -\mathrm{dE}\,\vec{i} = -\mathrm{dE}\sin\alpha\,\vec{i} \\ \overrightarrow{dE_y} = -\mathrm{dE}\,\vec{j} = -\mathrm{dE}\cos\alpha\,\vec{j} \end{cases}$$

Knowing that $\mathrm{dE} = \frac{k\,dq}{r^2} = \frac{k\,\lambda\,d\ell}{r^2}$



So $\begin{cases} dE_x = \frac{k\,\lambda\,d\ell}{r^2}\sin\alpha \\ dE_y = \frac{k\,\lambda\,d\ell}{r^2} = \cos\alpha \end{cases}$

Here we have three variable r, ℓ and α and we must choose only one variable, here we will choose α

$\cos\alpha = \frac{y}{r} \rightarrow r = \frac{y}{\cos\alpha}$

$\tan\alpha = \frac{\ell}{y} \rightarrow \ell = y\tan\alpha \rightarrow d\ell = \frac{y\,d\alpha}{\cos^2\alpha}$

when substituting r^2 and $d\ell$ by their values we get:

$$\begin{cases} dE_x = \frac{k\,\lambda\,d\ell}{r^2}\sin\alpha = \frac{k\,\lambda}{y}\sin\alpha\,d\alpha \\ dE_y = \frac{k\,\lambda\,d\ell}{r^2}\cos\alpha = \frac{k\,\lambda}{y}\cos\alpha\,d\alpha \end{cases}$$

$\Rightarrow E_x = \frac{k\,\lambda}{y}\int_{-\alpha_2}^{\alpha_1}\sin\alpha\,d\alpha = \frac{k\,\lambda}{y}(\cos\alpha_2 - \cos\alpha_1)$ and $E_y = \frac{k\,\lambda}{y}\int_{-\alpha_2}^{\alpha_1}\cos\alpha\,d\alpha = \frac{k\,\lambda}{y}(\sin\alpha_1 + \sin\alpha_2)$

$$\vec{E} = \frac{k\,\lambda}{y}(\cos\alpha_2 - \cos\alpha_1)\,\vec{i} + \frac{k\,\lambda}{y}(\sin\alpha_1 + \sin\alpha_2)\,\vec{j}$$

$$E = \sqrt{E_x{}^2 + E_y{}^2} = \frac{k\,\lambda}{y}\sqrt{2 + 2(\sin\alpha_1\sin\alpha_2 + \cos\alpha_1\cos\alpha_2)}$$

We have : cos(a+b) = cos(a)cos(b) – sin(a)sin(b) = -(sin(a)sin(b)-cos(a)cos(b))

$$E = \frac{k\,\lambda}{y}\sqrt{2 - 2\cos(\alpha_1 + \alpha_2)} = \frac{k\,\lambda}{y}\sqrt{2[1 - \cos(\alpha_1 + \alpha_2)]}$$

We also know that $1 - \cos(\alpha_1 + \alpha_2) = 2\sin^2\left(\frac{\alpha_1 + \alpha_2}{2}\right)$

$$\boxed{\mathbf{E} = \frac{2k\,\lambda}{y}\sin\left(\frac{\alpha_1 + \alpha_2}{2}\right)}$$

Another method consists of choosing the variable ℓ

$$\cos\alpha = \frac{y}{r} = \frac{y}{\sqrt{\ell^2+y^2}} \quad \text{and } \sin\alpha = \frac{\ell}{r} = \frac{\ell}{\sqrt{\ell^2+y^2}}$$

$$\begin{cases} dE_x = \dfrac{k\,\lambda.\ell.d\ell}{(\ell^2+y^2)^{\frac{3}{2}}} \\ dE_y = \dfrac{k\,\lambda\, y\, d\ell}{(\ell^2+y^2)^{\frac{3}{2}}} \end{cases}$$

$$\begin{cases} E_x = k\,\lambda \int_0^{2d} \dfrac{\ell.d\ell}{(\ell^2+y^2)^{\frac{3}{2}}} = k\,\lambda \left|\dfrac{-1}{\sqrt{\ell^2+y^2}}\right|_0^{2d} = k\,\lambda\left(\dfrac{-1}{\sqrt{4d^2+y^2}} + \dfrac{1}{y}\right) \\ E_y = k\,\lambda\, y\int_0^{2d} \dfrac{d\ell}{(\ell^2+y^2)^{\frac{3}{2}}} = k\,\lambda\, y\left|\dfrac{\ell}{y^2\sqrt{\ell^2+y^2}}\right|_0^{2d} = k\,\lambda\left(\dfrac{2k\,\lambda\, d}{y\sqrt{4d^2+y^2}}\right) \end{cases}$$

1)b) Let's calculate the potential created on a point M on the OY axis

$$dV = \frac{k\,dq}{r} = \frac{k\,\lambda\, d\ell}{(\ell^2+y^2)^{\frac{1}{2}}}$$

$$V = \int dV = k\,\lambda\int_0^{2d}\frac{d\ell}{(\ell^2+y^2)^{\frac{1}{2}}} = \left| k\,\lambda\ \ln\left(\ell + \sqrt{\ell^2+y^2}\right)\right|_0^{2d}$$

$$\boxed{\mathbf{V} = \boldsymbol{k\,\lambda}\left[\boldsymbol{ln}\left(\boldsymbol{2d} + \sqrt{\mathbf{4d^2} + \boldsymbol{y^2}}\right) - \boldsymbol{ln\ y}\right] = \boldsymbol{k\,\lambda}\ \frac{\boldsymbol{2d}+\sqrt{\mathbf{4d^2}+\boldsymbol{y^2}}}{\boldsymbol{y}}}$$

2)a) Let's calculate E where M is on the mediating plane of the wire AB

We have $\alpha_1 = \alpha_2 = \alpha$

$$E = \frac{2k\,\lambda}{y} sin\left(\frac{2\alpha}{2}\right) = \frac{2k\,\lambda}{y}\sin\alpha = \frac{2k\,\lambda}{y}\left(\frac{d}{r}\right)$$

$$\boxed{\mathbf{E} = \frac{\mathbf{2}\boldsymbol{k}\,\boldsymbol{\lambda}\,\mathbf{d}}{\boldsymbol{y}\sqrt{\mathbf{y^2}+\boldsymbol{d^2}}}}$$

2)b) We now deduce V where M is on the mediating plane of the wire AB

$$dV = \frac{k\,dq}{r} = \frac{k\,\lambda\, d\ell}{(\ell^2+y^2)^{\frac{1}{2}}}$$

$$\Rightarrow V = \int dV = k\,\lambda\int_{-d}^{d}\frac{d\ell}{(\ell^2+y^2)^{\frac{1}{2}}} = \left| k\,\lambda\ \ln\left(\ell + \sqrt{\ell^2+y^2}\right)\right|_{-d}^{d}$$

$$\boxed{\mathbf{V} = \boldsymbol{k\,\lambda}\,\mathbf{ln}\left(\frac{\boldsymbol{d}+\sqrt{\mathbf{d^2}+\boldsymbol{y^2}}}{-\boldsymbol{d}+\sqrt{\mathbf{d^2}+\boldsymbol{y^2}}}\right)}$$

3)b) Deduce the field E when the wire AB is of infinite length

When the wire AB is of infinite length that mean that $\alpha_1 = \alpha_2 = \frac{\pi}{2}$

$$E_x = \frac{k\,\lambda}{y}\,(\cos\alpha_2 - \cos\alpha_1) \quad = \frac{k\,\lambda}{y}\,(\cos\frac{\pi}{2} - \cos\frac{\pi}{2}) = 0$$

$$E_y = \frac{k\,\lambda}{y}\,(\sin\alpha_1 + \sin\alpha_2) \ = \frac{2\,k\,\lambda}{y}\sin\frac{\pi}{2} = \frac{2\,k\,\lambda}{y}$$

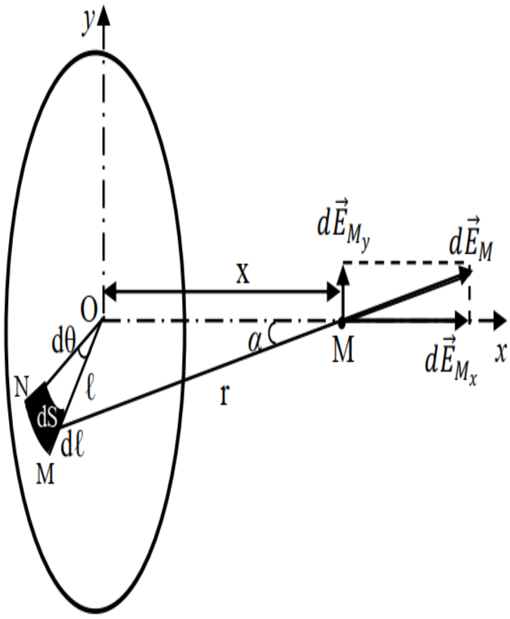
$$\boxed{\mathbf{E} = \boldsymbol{E_y} = \frac{\mathbf{2}\,\boldsymbol{k}\,\boldsymbol{\lambda}}{\boldsymbol{y}}}$$

<u>Exercise 2:</u>

1) Determination of the field EM created by the disk at point M of the axis OX, located at a distance x from the center O of the disk:

For reasons of symmetry with respect to the Ox axis,

the component $E_y = 0$, consequently the field E will only admit the component E_x



$dE_x = \text{dE} \cos\alpha$

$\text{dE} = \frac{k\,dq}{r^2} = \frac{k\,\sigma\,\text{dS}}{r^2}$

so $dE_x = \frac{k\,dq}{r^2}\cos\alpha$

$\text{dS} = MN\text{d}\ell = \ell\,\text{d}\theta\,\text{d}\ell$

$\cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{\ell^2 + x^2}}$

$$dE_x = k\,\sigma\,\frac{\ell\,\text{d}\theta\,\text{d}\ell}{\ell^2 + x^2}\,\frac{x}{\sqrt{\ell^2 + x^2}} = k\,\sigma\,\text{x}\left(\frac{\ell\,\text{d}\theta\,\text{d}\ell}{(\ell^2 + x^2)^{\frac{3}{2}}}\right)$$

$$E_x = k\,\sigma\,\text{x}\int_0^{2\pi}\text{d}\theta\int_0^R\frac{\ell\,\text{d}\ell}{(\ell^2 + x^2)^{\frac{3}{2}}} = 2\pi k\sigma x\left|\frac{-1}{\sqrt{\ell^2 + x^2}}\right|_0^R = 2\pi k\sigma x\left|\frac{-1}{\sqrt{\text{R}^2 + x^2}} + \frac{1}{x}\right|_0^R$$

In the end we get: $$\boxed{\mathbf{E} = \boldsymbol{E_x} = \boldsymbol{2\pi k\sigma}\left(\mathbf{1} - \frac{\boldsymbol{x}}{\sqrt{\mathbf{R}^2 + \boldsymbol{x}^2}}\right)}$$

1)b) Calculate the potential V created at a point M

$$\text{dV} = \frac{k\,dq}{r} = \frac{k\,\sigma\,\text{dS}}{r} = \frac{k\,\sigma\,\ell\,\text{d}\theta\,\text{d}\ell}{\sqrt{\ell^2 + x^2}} = k\,\sigma\int_0^{2\pi}\text{d}\theta\int_0^R\frac{\ell\,\text{d}\ell}{\sqrt{\ell^2 + x^2}} = 2\pi\,k\,\sigma\left|\sqrt{\ell^2 + x^2}\right|_0^R$$

$$\boxed{\mathbf{V} = \boldsymbol{2\pi\,k\,\sigma}\left(\sqrt{\mathbf{R}^2 + \boldsymbol{x}^2} - \boldsymbol{x}\right)}$$

2) Deduce the field E when the radius R tends toward infinity

When $\text{R} \to \infty \Rightarrow \text{E} = 2\pi\,k\,\sigma\left(1 - \frac{x}{\sqrt{\text{R}^2 + x^2}}\right) = 2\pi\,k\,\sigma\,(1 - 0) = 2\pi\,k\,\sigma$

$$\boxed{\mathbf{E} = \boldsymbol{2\pi\,k\,\sigma}}$$

It is the field of an infinite plane uniformly charged at a surface density σ > 0

<u>**Exercise 3:**</u>

1)a) The electrostatic potential V created in M by the 2 charges

$V(M) = V(+q) + V(-q) = \frac{k(+q)}{r} + \frac{k(-q)}{r}$

$V(M) = kq\frac{(r_2 - r_1)}{r_1 r_2}$

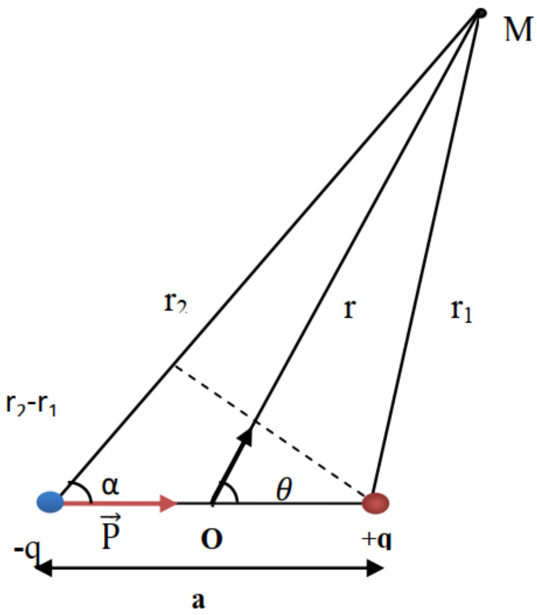
We have a≪r $\Rightarrow r_1 + r_2 \approx 2r$ *and* $r_1 r_2 \approx r^2$

And $r_2 - r_1 = a\cos\alpha$

A≪r $\Rightarrow \alpha \approx \theta$ so $(r_2 - r_1) = a\cos\theta$

$V(M) = V(r, \theta) = \frac{kqa\cos\theta}{r^2} = \frac{kP\cos\theta}{r^2}$

$$\boxed{\mathbf{V(M) = \frac{kP\cos\theta}{r^2}}}$$



1)b) Electric field of the dipole

Since V depends only on r and θ ,we will use polar coordinates to calculate the components of the electric field

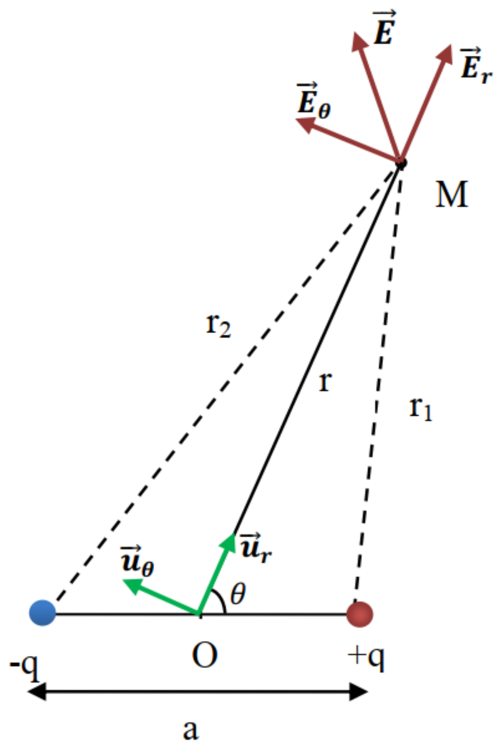
Let the polar reference frame be with center O and base vectors $(\vec{U}_r, \vec{U}_\theta)$

$\vec{E} = E_r\,\vec{U}_r + E_\theta\,\vec{U}_\theta$

And we have $\vec{E}(M) = -\overrightarrow{grad}(V)$

$\overrightarrow{grad}$ *in polar coordinates is* $\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\cdot\frac{\partial}{\partial\theta} \end{pmatrix}$

$$\begin{cases} E_r = -\frac{\partial V}{\partial r} = \frac{2kP\cos\theta}{r^3} \\ E_\theta = -\frac{1}{r}\cdot\frac{\partial V}{\partial\theta} = \frac{kP\sin\theta}{r^3} \end{cases}$$



The magnitude of the electric field is:

$E = \sqrt{E_r^{\ 2} + E_\theta^{\ 2}}$

$$\boxed{\mathbf{E = \frac{kP}{r^3}\sqrt{3\cos^2\theta + 1}}}$$

3) Equations of equipotential surfaces:

From the potential equation found previously, we can deduce the equation of equipotential surfaces:

$$V(M) = Cte = V_0 = \frac{kqa\cos\theta}{r^2} \Rightarrow r^2 = \frac{kqa\cos\theta}{V_0} = \frac{kp\cos\theta}{V_0}$$

$$r^2 = \frac{kp\cos\theta}{V_0}$$

Each value of V_0 corresponds to an equipotential surface located at distance r from O.